

SHAPE THEOREMS FOR POISSON HAIL ON A BIVARIATE GROUND

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ABSTRACT. We consider the extension of the Euclidean stochastic geometry Poisson Hail model to the case where the service speed is zero in some subset of the Euclidean space and infinity in the complement. We use and develop tools pertaining to sub additive ergodic theory in order to establish shape theorems for the growth of the ice-heap under light tail assumptions on the hailstone characteristics. The asymptotic shape depends on the statistics of the hailstones, the intensity of the underlying Poisson point process and on the geometrical properties of the zero speed set.

Keywords: Point process theory, Poisson rain, stochastic geometry, random closed set, time and space growth, shape, queuing theory, heaps, branching process, sub-additive ergodic theory.

1. INTRODUCTION

The present paper revisits the Poisson Hail queuing model introduced in [2]. This model features i.i.d. Random Closed Sets (hailstones) arriving on \mathbb{R}^d according to a Poisson rain. Each random closed set has a footprint, which is a Random Closed Set (RACS) of \mathbb{R}^d and a positive height. When the service speed of \mathbb{R}^d is equal to zero, the hailstones accumulate over time to form a random heap. The height of a tagged hailstone in this heap is the maximum of the heights of all hailstones that arrived before and that have a footprint that intersects that of the tagged one. This model was studied in [2]. It was shown that when the d 'th power of the random diameters and the random heights have light-tailed distributions, i.e. have finite exponential moment, then the growth of the random heap is asymptotically linear with time. This result was combined with a coupling from the past argument to show that when the service speed of \mathbb{R}^d is positive, then there exists a rain intensity below which the queuing dynamics is stable.

The present paper is focused on a generalization of the pure growth model of [2] to the case where the initial substrate is not the whole space but some subset of \mathbb{R}^d . Another way of looking at this generalization is to consider the following mix of speeds: some subset of \mathbb{R}^d has a service speed equal to zero, whereas the complement has speed equal to infinity. Hence, the hailstones whose footprint do not intersect that of any earlier hailstone

that is part of the current heap are served instantly. The others get aggregated to the heap. We show that under the above light tail assumptions, at any given time, the heap is a RACS of $\mathbb{R}^d \times \mathbb{R}$. The aim of the paper is to study the asymptotic shape of this RACS when time tends to infinity.

Two types of results are proved. The first type of results concerns the maximal height of the heap in some convex set of directions. These results generalize those on the growth speed of the random heap in [2]. The second type bear on the asymptotic shape of the footprint of the heap. Both types rely on Super Additive Ergodic Theory. The footprint shape theorem leverages the notion of gauge set which is introduced here for this purpose. This is done first for the model where is speed in infinite everywhere except for one point of \mathbb{R}^d . This is then extended to the case where the point with 0 speed is replaced by a finite set. The case of infinite 0 speed sets is analyzed too, with a focus on the case of cones.

This model belongs to the class of infinite dimensional (max,plus) linear systems. Among the few instances studied in the past, the closest is the work on infinite tandem queuing networks [1]. The underlying structure of the (max,plus) recursion in [1] is a two dimensional lattice. In contrast, here, the underlying structure of the recursion is random. Among common aspects, let us stress shape theorems. The lattice shape theorems in [1] are related to those in first passage percolation [7], in the theory of lattice animals [4, 5]. Those of the present paper pertain to first passage percolation in random media. This topic was studied in certain random graphs like the configuration model [3] lately. The shape theorems established in the present paper are based on random structures of the Euclidean space, which stem from point process theory (Poisson rain) and stochastic geometry (random closed sets).

2. THE MODEL

We consider a queue where the servers are the points of \mathbb{R}^d . We distinguish two types of servers: \mathcal{K} is the set servers with a service speed equal to zero, and $\mathbb{R}^d \setminus \mathcal{K}$ is that of servers with a service speed equal to infinity. The customers are characterized by:

- (1) A random closed set (RACS) of \mathbb{R}^d , such that the d -th power of the diameter has a light-tailed distribution;
- (2) A random service time also light-tailed.

These customers arrive to the queue (\mathbb{R}^d) according to a Poisson rain with intensity λ .

Starting with an empty queue at time $t = 0$, a customer gets queued if it hits \mathcal{K} or if it hits an earlier customer which was already queued.

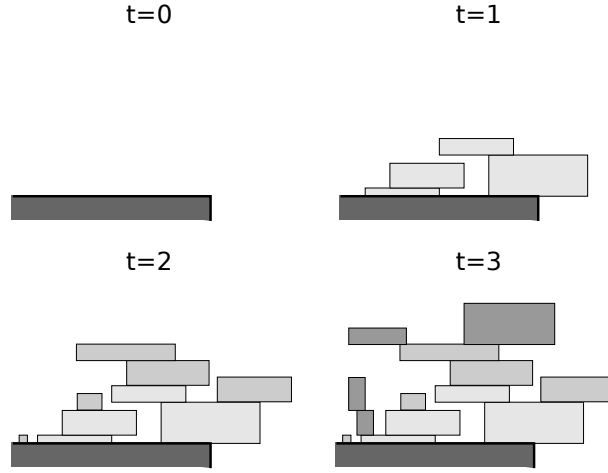


FIGURE 1. Evolution of a heap

The ice heap is a random set of $\mathbb{R}^d \times \mathbb{R}$, and the main questions of interest are about the growth of its height in various directions, and about the growth of its *spatial projection* (defined as its projection on \mathbb{R}^d), again in various directions.

2.1. Precise Formulation. Consider a homogeneous Poisson point process Φ in $\mathbb{R}^d \times \mathbb{R}$ with intensity λ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Φ can be seen as simple counting measure, namely as a sum of delta distributions at (different) points in $\mathbb{R}^d \times \mathbb{R}$. For every $A \subseteq \mathbb{R}^d \times \mathbb{R}$, $\Phi(A)$ counts the number of points that belong to the set A . By being Poisson homogeneous we mean the following:

- (1) $\Phi(A)$ has a Poisson distribution with parameter $\lambda|A|$, where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^d \times \mathbb{R}$.
- (2) Given pairwise disjoint subsets of $\mathbb{R}^d \times \mathbb{R}$, A_1, \dots, A_n , the random variables $\Phi(A_1), \dots, \Phi(A_n)$ are independent.

This point process is independently marked. Each point comes with a pair of marks. These pairs are independent and identically distributed. However stochastic dependence within a pair is allowed. Let $\{(C_{(x,t)}, \sigma_{(x,t)})\}_{(x,t) \in \Phi}$ denote the marks. These are i.i.d. random pairs. The mark of point (x, t) consists of a RACS $C_{(x,t)}$ centered at the origin and of a random variable $\sigma_{(x,t)}$ taking values in \mathbb{R}^+ .

Let

$$\xi_{(x,t)} = \text{diam}(C_{(x,t)}) := \sup\{|y - z|, y, z, \in C_{(x,t)}\}$$

be the diameter of set $C_{(x,t)}$. We assume that both random variables $\sigma_{(x,t)}$ and $\xi_{(x,t)}^d$ (the d th power of $\xi_{(x,t)}$) are *light-tailed*, namely such that

$$(2.1) \quad \mathbb{E}(\exp(c\xi_{(x,t)}^d)) < \infty, \quad \mathbb{E}(\exp(c\sigma_{(x,t)})) < \infty,$$

for some constant $c > 0$.

We say that *Assumption I* holds if, with a positive probability, the typical RACS C has a non-empty interior that includes the origin.

The homogeneity assumption is reflected by the following *compatibility property*. Given the group of translations

$$T_{(x_0,t_0)} : (x, t) \mapsto (x, t) + (x_0, t_0)$$

of $\mathbb{R}^d \times \mathbb{R}$, there exists $S : \mathbb{R}^d \times \mathbb{R} \times \Omega \rightarrow \Omega$ measurable and satisfying the following properties:

- (1) *Measure preserving*: For every $(x_0, t_0) \in \mathbb{R}^d \times \mathbb{R}$, $S_{(x_0,t_0)} : \Omega \rightarrow \Omega$ is measure preserving.
- (2) *Group property*: $S_{(x_0,t_0)} \circ S_{(x_1,t_1)} = S_{(x_0+x_1,t_0+t_1)}$ and $S_{(0,0)} = Id$.
- (3) *Compatibility*:

$$\Phi \circ S_{(x_0,t_0)}(A) = \Phi(T_{(x_0,t_0)}A).$$

One can then extend the sequence of marks to a random process $(C_{(x,t)}, \sigma_{(x,t)})$ defined on $\mathbb{R}^d \times \mathbb{R}$ and such that

$$(C_{(x,t)}, \sigma_{(x,t)}) = (C_{(0,0)}, \sigma_{(0,0)}) \circ S_{(x,t)}, \quad \forall (x, t).$$

Because of the Poisson and independence assumptions, there is no loss of generality in assuming that the flow S is ergodic. In particular, for every measurable $A \subseteq \Omega$ such that

$$\mathbb{P}(S_{(0,t)}^{-1}A\Delta A) = 0 \text{ for every } t \in \mathbb{R},$$

we have $\mathbb{P}(A) = 0$ or 1 . Here $B\Delta A = (B \setminus A) \cup (A \setminus B)$ is the symmetric difference of A and B .

2.2. Height Profile. Let $H_{(x,t)}$ be the height of the heap at location $x \in \mathbb{R}^d$ at time $t \geq 0$. We assume that initially

$$H_{(x,0)} = \begin{cases} 0 & \text{if } x \in \mathcal{K}, \\ -\infty & \text{if } x \notin \mathcal{K}. \end{cases}$$

Then, for $t > 0$, $H_{(x,t)}$ gets determined by the following formula. For $t > u \geq 0$,

$$(2.2) \quad H_{(x,t)} = H_{(x,u)} + \int_{[u,t)} \left(\sigma_{(x,v)} + \sup_{y \in C_{(x,v)} + x} (H_{(y,v)} - H_{(x,v)}) \right) N^x(dv),$$

where N^x denotes the Poisson point process of \mathbb{R} of RACS arrivals intersecting location x ,

$$N^x([a, b]) = \int_{\mathbb{R}^d \times [a, b]} 1(x \in C_{(y, s)} + y) \Phi(dy ds).$$

A similar construction of H is given in [2]. The only difference lies in the initial value which is $H(x, 0) = 0$, for all x in $[2]$.

2.3. Monotonicity. The proposed model is *monotone* in several arguments.

Monotonicity in \mathcal{K} . For two systems with the same data $(\Phi, \{C, \sigma\})$ but with initial substrates $\mathcal{K}^{(1)} \subseteq \mathcal{K}^{(2)}$, the associated heights $H^{(1)}$ and $H^{(2)}$ satisfy

$$H_{(x, t)}^{(1)} \leq H_{(x, t)}^{(2)} \text{ for every } (x, t) \in \mathbb{R}^d \times [0, \infty).$$

Similarly, there is monotonicity in t , in the σ 's and in the C 's.

3. THE STICK MODEL: $\mathcal{K} = \{0\}$

In this Section we consider the case $\mathcal{K} = \{0\}$ and call it *the stick model*. Theorem 3.1 shows that there exists a finite asymptotic limit for the maximal height of the associated heap $H_{(x, t)}$ (referred to as the *stick heap* below) in any given convex set of directions. Theorem 3.6 shows that there exists a finite asymptotic limit for how far the spatial projection of the heap grows, measured with respect to a *set-gauge* to be defined.

3.1. Height Growth. In this section, we focus the maximal height, $\mathbb{H}_t^{(\Theta)}$, of the stick heap among all directions in a set of directions Θ , which is defined as follows:

Definition 3.1. For $\Theta \subseteq S_+^d := \{(x, h) \in \mathbb{R}^d \times (0, 1] : |(x, h)| = 1\}$ non empty,

$$\mathbb{H}_t^{(\Theta)} := \sup \left\{ h \in [0, \infty) : \exists x \in \mathbb{R}^d \text{ such that } (x, h) \in |(x, h)|\Theta, H_{(x, t)} \geq h \right\}.$$

In particular, if $\Theta = \{(0, 1)\}$, the north pole of S_+^d , then $\mathbb{H}_t^{(\Theta)} = H_{(0, t)}$.

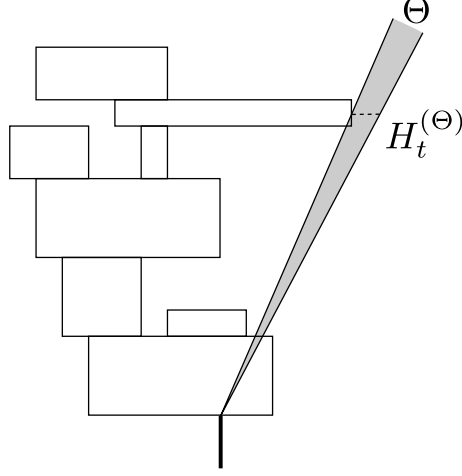
Since $H_{(0, t)} \geq 0$, the set where the supremum is evaluated in the last definition is non-empty as it always contains $h = 0$. This also implies that $\mathbb{H}_t^{(\Theta)} \geq 0$.

Definition 3.2. A set $\Theta \subseteq S_+^d$ is convex if for all $\theta_1, \theta_2 \in \Theta$ and $s \in [0, 1]$,

$$s\theta_1 + (1 - s)\theta_2 \in |s\theta_1 + (1 - s)\theta_2|\Theta.$$

Notice that if Θ is convex, then for all $a, b \geq 0$, we have

$$a\theta_1 + b\theta_2 \in |a\theta_1 + b\theta_2|\Theta.$$

FIGURE 2. Definition of $\mathbb{H}_t^{(\Theta)}$ with $\mathcal{K} = \{0\}$

Theorem 3.1. *For all $\Theta \subseteq S_+^d$ convex and closed, there exists a non-negative constant $\gamma^{(\Theta)}$ such that*

$$\lim_{t \rightarrow \infty} \frac{\mathbb{H}_t^{(\Theta)}}{t} = \lim_{t \rightarrow \infty} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta)}}{t} = \sup_{t > 0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta)}}{t} = \gamma^{(\Theta)} < \infty,$$

where the first limit holds both in the a.s. and the L_1 sense.

Before proving this theorem, we give a few preliminary lemmas.

The following lemma is a direct consequence of the independence of the Poisson rain in disjoint sets and of homogeneity. In this lemma, $\Phi \cap B$ denotes the set of points of Φ that belong to B .

Lemma 3.2. *Let $X : \Omega \rightarrow \mathbb{R}^d$ be a random variable which is independent of $\{\Phi \cap B, \{(C_{(y,s)}, \sigma_{(y,s)}), (y, s) \in B \cap \Phi\}; B \in \mathcal{B}(\mathbb{R}^d \times (0, \infty))\}$. Then, for every $\Theta \subseteq S_+^d$, the stochastic process $\{\mathbb{H}_t^{(\Theta)} \circ S_{(X,0)}, t > 0\}$ has the same law as $\{\mathbb{H}_t^{(\Theta)}, t > 0\}$ and it is independent of the σ -algebra generated by X and $\{\Phi \cap B, \{(C_{(y,s)}, \sigma_{(y,s)}), (y, s) \in B \cap \Phi\}; B \in \mathcal{B}(\mathbb{R}^d \times (-\infty, 0])\}$.*

Lemma 3.3.

$$(3.3) \quad \sup_{t > 0} \frac{\mathbb{E}\mathbb{H}_t^{(S_+^d)}}{t} < \infty.$$

The proof of Lemma 3.3 is quite close to that of Theorem 2 in [2]. In order to make the paper self-contained, we provide a proof in Appendix.

Lemma 3.4. *For all $0 \leq t_1 < t_2$ and $x, y \in \mathbb{R}^d$, the stick heap satisfies the following inequality:*

$$H_{(x+y, t_2)} \geq H_{(x, t_1)} + H_{(y, t_2 - t_1)} \circ S_{(x, t_1)}.$$

Proof. For $t \geq t_1$, let $\tilde{H}_{(z, t)}$ be constructed by (2.2) with the initial condition

$$\tilde{H}_{(z, t_1)} := \begin{cases} H_{x, t_1} & \text{if } z = x, \\ -\infty & \text{if } z \neq x. \end{cases}$$

Then, by monotonicity $H_{(x+y, t)} \geq \tilde{H}_{(x+y, t)}$ and it suffices to show that $\tilde{H}_{(x+y, t)} = H_{(x, t_1)} + H_{(y, t - t_1)} \circ S_{(x, t_1)}$. In order to do this we first verify that this relation holds for $t = t_1$, and we then use that fact that both sides satisfy (2.2) when $t \geq t_1$. Both steps are straightforward. \square

Proof of Theorem 3.1. Let $X_t \in \mathbb{R}^d$ be such that,

$$(X_t, \mathbb{H}_t^{(\Theta)}) \in |(X_t, \mathbb{H}_t^{(\Theta)})| \Theta, \quad H_{(X_t, t)} \geq \mathbb{H}_t^{(\Theta)}.$$

The existence of such an X_t is obtained from the proof of Corollary 1 in [6]. This proof shows that at time t , not only the height, but also the diameter of the heap is a.s. finite¹. Therefore, with probability 1, one can find at least one X_t that satisfies the above properties. There could be more than one and, in order for X_t to be a random variable (i.e. a measurable function), we may, for instance, take the smallest X_t in the lexicographical order.

For $0 \leq t_1 \leq t_2$, let

$$\mathbb{H}_{t_1, t_2}^{(\Theta)} := \mathbb{H}_{t_1 - t_2}^{(\Theta)} \circ S_{(X_{t_1}, t_1)}.$$

In order to prove that the limit in the theorem exists and is a.s. constant, we use the Super-additive Ergodic Theorem of Liggett, see [6]. We have to verify that the following properties hold:

- (1) Super-additivity: For $t_2 > t_1 \geq 0$

$$\mathbb{H}_{t_2}^{(\Theta)} \geq \mathbb{H}_{t_1}^{(\Theta)} + \mathbb{H}_{t_1, t_2}^{(\Theta)}.$$

- (2) For $t_2 > t_1 \geq 0$, the joint distribution of $\{\mathbb{H}_{t_2, t_2+k}^{(\Theta)}, k > 0\}$ is the same as that of $\{\mathbb{H}_{t_1, t_1+k}^{(\Theta)}, k > 0\}$.

- (3) For $k > 0$, $\{\mathbb{H}_{nk, (n+1)k}^{(\Theta)}, n > 0\}$ is a stationary process.

- (4) Bound for the expectation:

$$\sup_{t > 0} \frac{\mathbb{E} \mathbb{H}_t^{(\Theta)}}{t} < \infty.$$

¹Later on we will also prove an upper bound for this diameter in Lemma 3.5

To prove (1), let $t_2 > t_1 \geq 0$ be fixed and let

$$V = \{(x, h) \in \mathbb{R}^d \times (0, \infty) : (x, h) \in |(x, h)|\Theta, h \leq H_{(x, t_2 - t_1)} \circ S_{(X_{t_1}, t_1)}\}$$

For $(x, h) \in V$ we have by the convexity of Θ that,

$$(3.4) \quad (X_{t_1} + x, \mathbb{H}_{t_1}^{(\Theta)} + h) \in |(X_{t_1} + x, \mathbb{H}_{t_1}^{(\Theta)} + h)|\Theta.$$

Moreover,

$$(3.5) \quad \begin{aligned} \mathbb{H}_{t_1}^{(\Theta)} + h &\leq H_{(X_{t_1}, t_1)} + H_{(x, t_2 - t_1)} \circ S_{(X_{t_1}, t_1)}, \\ &\leq H_{(X_{t_1} + x, t_2)}, \end{aligned}$$

where we used Lemma 3.4 in the last inequality. By combining (3.4) and (3.5) we get that $\mathbb{H}_{t_2}^{(\Theta)} \geq \mathbb{H}_{t_1}^{(\Theta)} + h$, which implies the super-additive inequality after taking the supremum of h over $(x, h) \in V$.

To prove (2) we go back to the definition of $\mathbb{H}_{t_i, t_i + k}^{(\Theta)}$ for $i = 1, 2$,

$$\{\mathbb{H}_{t_i, t_i + k}^{(\Theta)}, k > 0\} = \{\mathbb{H}_k^{(\Theta)} \circ S_{(X_{t_i}, t_i)}, k > 0\}.$$

By Lemma 3.2 we get that both families of random variables have the same joint distribution as $\{\mathbb{H}_k^{(\Theta)}, k > 0\}$.

To prove (3) it is enough to check that, for $k > 0$ fixed, the random variables $\{\mathbb{H}_{nk, (n+1)k}^{(\Theta)}, n > 0\}$ are identically distributed and independent. By definition,

$$\mathbb{H}_{nk, (n+1)k}^{(\Theta)} = \mathbb{H}_k^{(\Theta)} \circ S_{(X_{nk}, nk)}.$$

Using Lemma 3.2 once again, we get that $\mathbb{H}_{nk, (n+1)k}^{(\Theta)}$ is distributed as $\mathbb{H}_k^{(\Theta)}$. Then the independence property follows again from Lemma 3.2.

Finally, (4) results from the upper bound given by Lemma 3.3. \square

3.2. Spatial Projection.

Definition 3.3. For $t \geq 0$, let F_t be the spatial projection of the heap, namely the RACS of \mathbb{R}^d which is the union of all the RACS added to the heap up to time t :

$$F_t := \{x \in \mathbb{R}^d : H_{(x, t)} \geq 0\}.$$

If the sets $C(x, t)$ are a.s. connected, so is F_t . However, if the sets $C(x, t)$ are a.s. convex, F_t has no reason to be convex.

In general, F_t is not necessarily a RACS. However under the light-tailedness assumptions (2.1):

Lemma 3.5. For all finite t , F_t is a RACS and

$$(3.6) \quad \sup_{t > 0} \frac{\mathbb{E}(\text{diam}(F_t))}{t} < \infty.$$

Proof. The proof is an application of Lemma 3.3, which follows the ideas in the proof of Corollary 1 in [2].

The fact that F_t is a RACS follows from the the upper bound branching process constructed for F_t in the proof of Lemma 3.3. This branching process has a.s. finitely many offspring in each generation. This implies that all finite $t > 0$, only a finite number of RACS $C_{(x,s)}$ may contribute to F_t .

We now prove (3.6). First notice that the set F_t does not depend on the heights. However we will make use of them in the following way. Assume $\sigma_{(x,t)} = \xi_{(x,t)} = \text{diam}(C_{(x,t)})$. We now show that under this assumption,

$$4 \sup_{x \in \mathbb{R}^d} H_{(x,t)} \geq \text{diam}(F_t).$$

For every $x \in \mathbb{R}^d$ such that $H_{(x,t)} \geq 0$, there exists an integer n and some set of points $(x_1, t_1), \dots, (x_n, t_n) \in \mathbb{R}^d \times [0, t)$ such that:

- (1) $(x_i, t_i) \in \text{supp } \Phi$ for $i = 1, \dots, n$;
- (2) $0 \leq t_i < t_{i+1} < t$ for $i = 1, \dots, (n-1)$;
- (3) $x \in C_{(x_n, t_n)}$ and $H_{(x,s)} = H_{(x, t_n)}$ for $s \in [t_n, t)$;
- (4) for $i = 1, \dots, (n-1)$, there exists $y_i \in C_{(x_{i+1}, t_{i+1})} \cap C_{(x_i, t_i)}$ such that $H_{(y_i, s)} = H_{(y_i, t_i)}$ for $s \in [t_i, t_{i+1})$;
- (5) $0 \in C_{(x_1, t_1)}$ and $H_{(0,s)} = 0$ for $s \in [0, t_1)$.

Therefore,

$$|x| \leq |x - x_n| + \sum_{i=1}^{n-1} |x_{i+1} - x_i| + |x_1| \leq 2 \sum_{i=1}^n \text{diam}(C_{(x_i, t_i)}) = 2H_{(x,t)}.$$

Maximizing over $\{x \in \mathbb{R}^d : H_{(x,t)} \geq 0\}$ and applying Lemma 3.3 concludes the proof. \square

Definition 3.4. Given a direction $v \in S^{d-1}$ and a closed set $A \subseteq \mathbb{R}^d$, containing the origin, let

$$D_t^{(A,v)} := \inf \{r \in [0, \infty) : (A + rv) \cap F_t = \emptyset\}.$$

where the infimum of an empty set is ∞ .

Here are a few examples: If $A = \{0\}$ then $D_t^{(A,v)}$ can be interpreted as the *internal growth* of F_t in the v direction at time t . It is also the contact distance with free space in the v -direction. Other interesting cases arise when $A = \{x \in \mathbb{R}^d : x \cdot v \geq 0\}$ or $A = \{x \in \mathbb{R}^d : x = \alpha v, \alpha \geq 0\}$; then $D_t^{(A,v)}$ can be interpreted as the *external growth* of F_t in the v direction. These cases are covered in Theorem 3.6 and illustrated in Figure 3.

Definition 3.5. The pair (v, A) , where $v \in S^{d-1}$ is a direction and $A \subseteq \mathbb{R}^d$ a closed set, forms a set-gauge if

- (1) A contains the origin and for every $a \in A$, $A + a \subseteq A$,

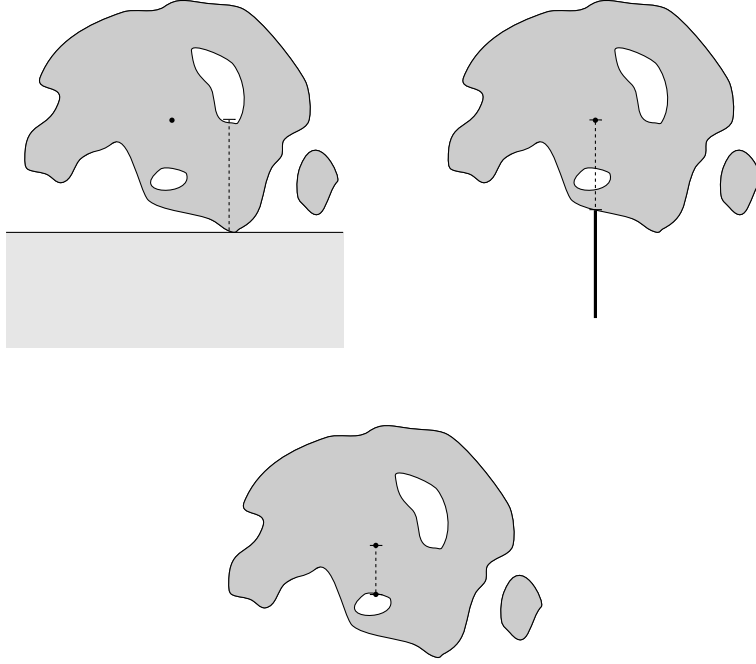


FIGURE 3. Different set-gauges measuring the spatial growth of F_t . The direction of v is South. The top-left case is $A = \{x \in \mathbb{R}^d : x \cdot v \geq 0\}$; the top-right case is $A = \{x \in \mathbb{R}^d : x = \alpha v, \alpha \geq 0\}$; the bottom case is $A = \{0\}$.

(2) $-v$ does not belong to the convex hull of A .

The three above examples are set-gauges. Here are other examples: If A is a closed convex cone of \mathbb{R}^d , different from \mathbb{R}^d , and $-v \notin A$, then (v, A) forms a set-gauge.

If (v, A) forms a set-gauge, then (v, B) , where $B := \bigcup_{r>0} (A + rv)$ also form a set-gauge. In this case

$$D_t^{(B,v)} = \sup \{r \in [0, \infty) : (A + rv) \cap F_t \neq \emptyset\}.$$

Note that for all set-gauges (v, A) , $D_t^{(A,v)}$ is a.s. finite. This follows from the property that F_t is a a.s. compact and the assumption that $-v$ does not belong to the convex hull of A .

Our main result is:

Theorem 3.6. *Given a direction $v \in S^{d-1}$ and a closed set $A \subseteq \mathbb{R}^d$, such that (v, A) forms a set-gauge, there exists a non negative constant $\phi = \phi^{A,v}$ such that*

$$\lim_{t \rightarrow \infty} \frac{D_t^{(A,v)}}{t} = \lim_{t \rightarrow \infty} \frac{\mathbb{E} D_t^{(A,v)}}{t} = \sup_{t>0} \frac{\mathbb{E} D_t^{(A,v)}}{t} =: \phi < \infty,$$

where the first limit is both a.s. and in L_1 .

Proof. Once again the proof relies on the distributional Super-additive Ergodic Theorem. Let $X_t \in \mathbb{R}^d$ be a random variable such that,

$$X_t \in (A + D_t^{(A,v)}v) \cap F_t.$$

The existence of a finite X_t satisfying this relation follows from the fact that both A and F_t are compact. It also uses the fact that $-v$ does not belong to the convex hull of A . There is no reason to have uniqueness. However, we can use the same construction as in the proof of Theorem 3.1 to cope with multiple solutions.

For $0 \leq t_1 \leq t_2$, let

$$D_{t_1, t_2}^{(A,v)} := D_{t_2 - t_1}^{(A,v)} \circ S_{X_{t_1}, t_1}.$$

By Lemma 3.2, properties analogous to properties (2) and (3) in the proof of Theorem 3.1 do hold. We now prove the super-additivity and the boundedness of the expectations.

In order to prove the super-additive inequality, it is enough to show that, for every $r < D_{t_1}^{(A,v)} + D_{t_1, t_2}^{(A,v)}$,

$$(3.7) \quad (A + rv) \cap F_{t_2} \neq \emptyset.$$

If $r < D_{t_1}^{(A,v)}$, this follows from the monotonicity of F_t w.r.t. time and from the definition of $D_{t_1}^{(A,v)}$. Let now $r = D_{t_1}^{(A,v)} + r'$, with $r' \in [0, D_{t_1, t_2}^{(A,v)}]$. From the definition of F_t ,

$$F_{t_2 - t_1} \circ S_{X_{t_1}, t_1} + X_{t_1} \subseteq F_{t_2}.$$

From the definition of a set-gauge and the property $X_{t_1} \in A + D_{t_1}^{(A,v)}v$,

$$A + X_{t_1} + r'v \subseteq A + rv.$$

From the definition of $D_{t_1, t_2}^{(A,v)}$, for $r' < D_{t_1, t_2}^{(A,v)}$,

$$(A + r'v) \cap (F_{t_2 - t_1} \circ S_{X_{t_1}, t_1}) \neq \emptyset,$$

which implies

$$(A + X_{t_1} + r'v) \cap (F_{t_2 - t_1} \circ S_{X_{t_1}, t_1} + X_{t_1}) \neq \emptyset$$

and (3.7) follows from the last two inclusions.

Now we prove boundedness of expectations. Given that A and v form a gauge there exists a hyperplane given by $P = \{x \in \mathbb{R}^d : x \cdot w = 0\}$, with $w \in S^{d-1}$, that separates $-v$ and A , i.e.

- (1) $v \cdot w > 0$,
- (2) $a \cdot w \geq 0$ for every $a \in A$.

Then, letting $A' = \{x \in \mathbb{R}^d : x \cdot w \geq 0\}$, and using the monotonicity inherited from the fact that $A \subseteq A'$,

$$D_t^{(A,v)} \leq D_t^{(A',w)} \leq \frac{\text{diam}(F_t)}{|v \cdot w|}.$$

Finally, applying Lemma 3.5 we get

$$\limsup_{t \rightarrow \infty} \frac{\mathbb{E} D_t^{(A,v)}}{t} \leq \frac{\limsup_{t \rightarrow \infty} \mathbb{E} \text{diam}(F_t)/t}{|v \cdot w|} < \infty.$$

So the proof is complete. \square

Now we focus on the gauges with $A = \{0\}$. Our aim is to prove that, under an extra assumption on the C RACS,

$$\lim_{t \rightarrow \infty} \frac{D_t^{(A,v)}}{t} > 0.$$

This will in turn imply that, for every $x \in \mathbb{R}^d$, the time that it takes for F_t to hit x is a.s. finite.

Lemma 3.7. *Assume that the intensity of Φ is positive and that, with a positive probability, the typical RACS C has a non-empty interior that contains the origin. Then, for $A = \{0\}$ and $v \in S^{d-1}$, we have*

$$\lim_{t \rightarrow \infty} \frac{D_t^{(A,v)}}{t} > 0.$$

Proof. By Theorem 3.6,

$$\lim_{t \rightarrow \infty} \frac{D_t^{(A,v)}}{t} = \sup_{t > 0} \frac{\mathbb{E} D_t^{(A,v)}}{t} \geq \mathbb{E} D_1^{(A,v)}.$$

From the lemma conditions, there is a positive r such that, with positive probability, C contains the ball B_r with radius r centered at the origin. By the thinning property of the Poisson point process, we may consider only the Poisson rain (with a smaller, but positive intensity) with RACS that include B_r . Then, using the monotonicity mentioned earlier, we may take $C = B_r$. In the latter case, it is not difficult to see that

$$\mathbb{P}(D_1^{(A,v)} > r/2) \geq \mathbb{P}(\Phi(B_{r/2} \times (0, 1]) > 0) > 0.$$

Then $\mathbb{E} D_1^{(A,v)} > 0$, and the result follows. \square

Definition 3.6. *Given a set $K \in \mathbb{R}^d$ let $\tau(K)$ denote the time it takes for F_t to cover K , i.e.*

$$\tau(K) := \inf\{t \in [0, \infty] : K \setminus F_t = \emptyset\}.$$

Remark 3.8. $\tau(K)$ is a stopping time in the sense that $\{\tau(K) \leq t\}$ belongs to the σ -algebra generated by

$$\{\Phi \cap B, \{(C_{(y,s)}, \sigma_{(y,s)}), (y, s) \in \Phi \cap B\}, B \in \mathcal{B}(\mathbb{R}^d \times [0, t])\}.$$

In particular, for all random variables X measurable w.r.t. the σ -algebra generated by

$$\{\Phi \cap B, \{(C_{(y,s)}, \sigma_{(y,s)}), (y, s) \in \Phi \cap B\}, B \in \mathcal{B}(\mathbb{R}^d \times [0, \infty))\},$$

for all $x \in \mathbb{R}^d$ and $M \geq 0$, the random variables $1_{\{\tau \leq M\}}$ and $X \circ S_{(x,M)}$ are independent.

Corollary 3.9. Assume that the intensity of Φ is positive and that Assumption I holds. Then, for all bounded sets $K \subseteq \mathbb{R}^d$, $\tau(K)$ is a.s. finite.

Proof. It suffices to assume, by the same reasoning as in the previous proof, that the typical C is deterministic and given by balls of some sufficiently small radius $r > 0$. Let $\{x_1, \dots, x_n\} \subseteq K \setminus \{0\}$ such that,

$$K \subseteq \bigcup_{i=1}^n B_{r/2}(x_i).$$

Denote also $v_i = \frac{x_i}{|x_i|}$ for $i = 1, \dots, n$.

Consider now $C' := B_{r/2}$ and $F'_t, D_t^{(A,v)}$ constructed from C' and $A = \{0\}$. By Lemma 3.7 we have that for $i = 1, \dots, n$,

$$\tau'_i := \inf\{t \in [0, \infty] : D_t^{(A,v_i)} \geq |x_i|\} < \infty, \quad \text{a.s.}$$

By construction of C' , if $x_i \in F'_t$, then also $B_{r/2}(x_i) \subseteq F_t$, which concludes the proof. \square

3.3. Phase Transition. From Theorem 3.1, for all $\theta \in S_+^d$, there exists a growth rate in direction v_θ , which will be denoted by γ_θ .

Let us represent the vector v_θ as

$$v_\theta = \sin(\phi)e_{d+1} + \cos(\phi)w,$$

with e_{d+1} corresponding to the time dimension and w a unit vector of the vectorial space (e_1, \dots, e_d) associated with the spatial dimension. For fixed $w \in S^{d-1}$, there is a bijection between $\theta \in S_+^d$ and $\phi \in [0, \pi/2]$. The mapping $\phi \rightarrow \theta$ will be denoted by θ_w . The function $\gamma_{\theta_w(\phi)}$ admits the following phase transition:

Theorem 3.10. Under Assumption I, for all $w \in \mathbb{R}^d$, there exists an angle, $0 < \phi_*(w) < \pi/2$ such that $\gamma_{\theta_w(\phi)}$ is positive for any $\phi \in (\phi_*(w), \pi/2]$ and $\gamma_{\theta_w(\phi)} = 0$ for any $\phi \in [0, \phi_*(w))$.

Proof. Let us first show that if $\gamma_{\theta_w(\phi)} > 0$, then for all $\phi < \hat{\phi} < \frac{\pi}{2}$, $\gamma_{\theta_w(\hat{\phi})} > 0$. For any $t > 0$, let x_t be defined by

$$t = x_t \tan \phi$$

and let

$$\hat{t} = x_t \tan \hat{\phi}.$$

Then

$$H_{(x_t w, \hat{t})} \geq H_{(x_t w, t)} + H_{(0, \hat{t}-t)} \circ S_{(x_t w, t)},$$

so that

$$\gamma_{\theta_w(\hat{\phi})} = \lim_{\hat{t}} \frac{H_{(x_t w, \hat{t})}}{\hat{t}} \geq \gamma_{\theta_w(\phi)} \frac{\tan \phi}{\tan \hat{\phi}} > 0.$$

Let now

$$\phi_*(w) = \inf\{\phi \in [0, \pi/2] : \gamma_{\theta_w(\phi)} > 0\}.$$

If $\phi = \pi/2$, then $\gamma_{\theta_w(\phi)} > 0$. Hence, $\phi_*(w)$ is well defined. It follows from the last monotonicity property that it is the threshold above which $\gamma_{\theta_w(\phi)} > 0$.

It remains to prove that this threshold is non degenerate.

Let us first prove that it is positive. Let s_w denote the spatial growth rate in direction w and h the vertical growth rate. Both s_w and h are positive and finite. So if the angle ϕ is smaller than $\arctan(\frac{h}{s_w}) > 0$, then $\gamma_{\theta_w(\phi)} = 0$.

Let us now show that $\phi_*(w) < \pi/2$. For all $n \in \mathbb{N}$, let $x_n = \frac{nr}{2}w$ (here r is the radius of the ball the existence of which is stated in Assumption I). Let Π_n be the Poisson rain of RACS's that contain a ball of radius r centered at x_n . Let $t_0 \equiv T_0 > 0$ be the first time of arrival of a RACS of Π_0 and, for each $n = 0, 1, \dots$, let $T_{n+1} = T_n + t_{n+1}$ be the first arrival time after T_n of a RACS of Π_{n+1} . The random variables t_n are i.i.d. exponential with mean, say, $b > 0$. Also, $H_{(x_n, T_n)}$ is not smaller than the sum of $(n+1)$ i.i.d. random variables with distribution $H_{(0, T_0)}$. Since $\mathbb{E}H_{(0, T_0)} > 0$, it follows that $\liminf H_{(x_i, T_i)}/T_i > 0$ a.s. Further, $T_i/x_i \rightarrow 2b/r < \infty$, so $\phi_*(w) \leq \arctan(2b/r) < \pi/2$. \square

4. THE MODEL WITH \mathcal{K} BOUNDED

In this section we study the growth of the heap starting with \mathcal{K} , in some convex set of directions Θ measured from the origin of \mathbb{R}^d , when $\mathcal{K} \subseteq \mathbb{R}^d$ is a closed and bounded initial substrate. We consider two cases, depending on whether the point where the origin belongs to \mathcal{K} or not.

4.1. Asymptote at $0 \in \mathcal{K}$. In this section we fix $0 \in \mathcal{K}$. Let $\mathcal{K}^{(0)} = \{0\}$. Whenever $\mathbb{H}_t^{(\Theta)}$ is computed with respect to $\mathcal{K}^{(0)}$ (resp. \mathcal{K}) we denote it by $\mathbb{H}_t^{(\Theta, 0)}$ (resp. $\mathbb{H}_t^{(\Theta)}$). An analogous notation is used for all the other possible constructions. Given a constant $M \geq 0$, the measure preserving transformation $S_{(0, M)}$ of Ω to itself is denoted by S_M .

In the next lemma $\tau := \tau(\mathcal{K})$ denotes the time it takes for $F_t^{(0)}$ to cover the whole set \mathcal{K} .

Lemma 4.1. *For all $\Theta \subseteq S_+^d$ closed and for all $M, t \geq 0$, the following inequalities hold on $\{\tau \leq M\}$,*

$$\mathbb{H}_{M+t}^{(\Theta,0)} \geq \mathbb{H}_t^{(\Theta)} \circ S_M \geq \mathbb{H}_t^{(\Theta,0)} \circ S_M.$$

Proof. On $\tau \leq M$, $\mathcal{K} \subseteq F_M^{(0)}$, so that for every $x \in \mathbb{R}^d$, $H_{(x,M)}^{(0)} \geq H_{(x,0)} \circ S_M$. This implies that $H_{(x,M+t)}^{(0)} \geq H_{(x,t)} \circ S_M$ for all $t > 0$ by the monotonicity in the construction of H . The left-most inequality then follows.

The right-most inequality is just a consequence of the monotonicity w.r.t. the initial substrates $\mathcal{K}^{(0)} \subseteq \mathcal{K}$. \square

Lemma 4.2. *Under the assumptions of Corollary 3.9 and Theorem 3.1, if $0 \in \mathcal{K}$, we have*

$$\lim_{t \rightarrow \infty} \frac{\mathbb{H}_t^{(\Theta)}}{t} = \lim_{t \rightarrow \infty} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta)}}{t} = \sup_{t > 0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t} = \gamma^{(\Theta)} < \infty,$$

where the first limit is both in the a.s. and the L_1 sense.

Notice that the rightmost term is the one corresponding to $\mathcal{K}^{(0)}$; this tells us that asymptotically, the heaps starting at \mathcal{K} or $\mathcal{K}^{(0)}$ behave similarly in terms of directional shape.

Proof. By Lemma 4.1, Theorem 3.1 and the fact that S_M is measure preserving, we have

$$\lim_{t \rightarrow \infty} \frac{\mathbb{H}_t^{(\Theta)} \circ S_M}{t} 1_{\{\tau \leq M\}} = \sup_{t > 0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t} 1_{\{\tau \leq M\}} \text{ a.s.}$$

Hence

$$\begin{aligned} \mathbb{P} \left(\lim_{t \rightarrow \infty} \frac{\mathbb{H}_t^{(\Theta)}}{t} = \sup_{t > 0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t} \right) &= \mathbb{P} \left(\lim_{t \rightarrow \infty} \frac{\mathbb{H}_t^{(\Theta)} \circ S_M}{t} = \sup_{t > 0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t} \right) \\ &\geq \mathbb{P}(\tau \leq M). \end{aligned}$$

Since $M > 0$ is arbitrary and τ is finite a.s., we get the a.s. convergence of $\mathbb{H}_t^{(\Theta)}/t$ to the announced limit.

Now we proceed to show the convergence in L_1 . By Lemma 4.1,

$$\left(\frac{\mathbb{H}_{M+t}^{(\Theta,0)}}{t} - \sup_{t > 0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t} \right) 1_{\{\tau \leq M\}} \geq \left(\frac{\mathbb{H}_t^{(\Theta)} \circ S_M}{t} - \sup_{t > 0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t} \right) 1_{\{\tau \leq M\}}.$$

Then,

$$\mathbb{E} \left| \frac{\mathbb{H}_{M+t}^{(\Theta,0)}}{t} - \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t} \right| \geq \mathbb{E} \left| \left(\frac{\mathbb{H}_t^{(\Theta)} \circ S_M}{t} - \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t} \right) 1_{\{\tau \leq M\}} \right|.$$

By the independence property given in Remark 3.8, and using again that S_M is measure preserving,

$$\mathbb{E} \left| \frac{\mathbb{H}_{M+t}^{(\Theta,0)}}{t} - \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t} \right| \geq \mathbb{E} \left| \frac{\mathbb{H}_t^{(\Theta)}}{t} - \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t} \right| \mathbb{P}(\tau \leq M).$$

Choose M sufficiently large so $\mathbb{P}(\tau \leq M) > 0$. Then letting $t \rightarrow \infty$ concludes the proof. \square

4.2. Asymptote at $0 \notin \mathcal{K}$. In this section we assume that $0 \notin \mathcal{K}$ and that \mathcal{K} is non empty. We use the following notation: $\mathcal{K}^{(0)} = \{0\}$ and $\mathcal{K}^{(1)} = \mathcal{K} \cup \{0\}$. Whenever $\mathbb{H}_t^{(\Theta)}$ is computed with respect to $\mathcal{K}^{(0)}$, (resp. $\mathcal{K}^{(1)}$ or \mathcal{K}) we denote it by $\mathbb{H}_t^{(\Theta,0)}$ (resp. $\mathbb{H}_t^{(\Theta,1)}$ or $\mathbb{H}_t^{(\Theta)}$). We use analogous notation for all other possible constructions.

In the next lemma $\tau := \tau(\mathcal{K})$ is the time it takes for F_t to hit the origin.

Lemma 4.3. *For all closed $\Theta \subseteq S_+^d$ and all $M, t \geq 0$, the following inequalities hold on $\{\tau \leq M\}$,*

$$\mathbb{H}_{t+M}^{(\Theta,1)} \geq \mathbb{H}_{t+M}^{(\Theta)} \geq \mathbb{H}_t^{(\Theta,0)} \circ S_M.$$

Proof. The proof is very similar to that of Lemma 4.1. On $\tau \leq M$, $\mathcal{K}^{(0)} \subseteq F_M$; therefore for every $x \in \mathbb{R}^d$, $H_{(M,x)} \geq H_{(0,x)}^{(0)} \circ S_M$, which implies the second inequality. The first inequality is a consequence of the monotonicity. \square

Using this lemma (instead of Lemma 4.1) and the the same ideas as in the proof of Lemma 4.2 gives:

Theorem 4.4. *Under the assumptions of Corollary 3.9 and Theorem 3.1, in all cases ($0 \in \mathcal{K}$ or $0 \notin \mathcal{K}$),*

$$\lim_{t \rightarrow \infty} \frac{\mathbb{H}_t^{(\Theta)}}{t} = \lim_{t \rightarrow \infty} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta)}}{t} = \sup_{t>0} \frac{\mathbb{E}\mathbb{H}_t^{(\Theta,0)}}{t} = \gamma^{(\Theta)} < \infty,$$

with the first limit holding both a.s. and in the L_1 sense.

5. THE MODEL WITH \mathcal{K} A CONVEX CONE AND ITS GENERALIZATIONS

In this section, the substrate is first a convex cone of \mathbb{R}^d with its vertex at the origin, and then an object similar to such a cone but more general.

Definition 5.1. Given $\mathcal{C} \subseteq \mathbb{R}^d$ a closed convex cone with vertex at the origin, we define $\Theta(\mathcal{C}) \subseteq S_+^d$ to be the following subset of S_+^d :

$$\Theta(\mathcal{C}) := S_+^d \cap (\mathcal{C} \times \mathbb{R}).$$

For the proofs of this section, we use yet another property of the model, which is some form of invariance by time reversal. Consider the reflection

$$R : (x, t) \mapsto (x, -t).$$

Because the Poisson rain is invariant in law by R , and because the marks are i.i.d., there exists a measure preserving $V : \Omega \rightarrow \Omega$ which is compatible with R , i.e.:

$$(\Phi \circ V)(A) = \Phi(RA) \quad C_{(x,t)} \circ V = C_{R(x,t)}, \quad \sigma_{(x,t)} \circ V = \sigma_{R(x,t)}.$$

In the following theorem, $\mathbb{H}_t^{(\Theta,0)}$ and $\mathbb{H}_t^{(\Theta,1)}$ are the heights computed when starting with the substrate $\mathcal{K}^{(0)} := \{0\}$ or $\mathcal{K}^{(1)} := \{0, x\}$ respectively whereas $H_{(x,t)}$ is the height at x when starting with the substrate $\mathcal{K} = \mathcal{C}$.

Theorem 5.1. Under the assumptions of Corollary 3.9, for all closed convex cones $\mathcal{K} \subseteq \mathbb{R}^d$ and all $x \in \mathbb{R}^d$,

(5.8)

$$\lim_{t \rightarrow \infty} \frac{\max(0, H_{(x,t)})}{t} = \lim_{t \rightarrow \infty} \frac{\mathbb{E} \max(0, H_{(x,t)})}{t} = \sup_{t > 0} \frac{\mathbb{E} \mathbb{H}_t^{(\Theta(\mathcal{K}),0)}}{t} = Z < \infty,$$

where the first limit is in L_1 ²

Proof. Case 1: x is the vertex of the cone.

Without loss of generality, the vertex is assumed to be at the origin. The key observation is the following duality between the dynamics starting with \mathcal{K} and $\mathcal{K}^{(0)}$,

$$H_{(0,t)} = \mathbb{H}_{(0,t)}^{(\Theta(\mathcal{K}),0)} \circ V \circ S_{(0,t)}.$$

Once this gets established, the L_1 limit results from the fact that $S_{(0,t)} \circ V$ is measure preserving and therefore both sides are equivalent in distribution.

We first prove that $H_{(0,t)} \leq \mathbb{H}_{(0,t)}^{(\Theta(\mathcal{K}),0)} \circ V \circ S_{(0,t)}$. Consider the set of points $(x_0, t_0), \dots, (x_n, t_n) \in \mathbb{R}^d \times [0, t]$ "connecting" 0 with its height at t . Specifically, these satisfy:

- (1) $(x_i, t_i) \in \text{supp } \Phi$ for $i = 0, \dots, n$.
- (2) $0 \leq t_i < t_{i+1} < t$ for $i = 0, \dots, (n-1)$.
- (3) $0 \in C_{(x_n, t_n)}$ and $H_{(0,s)} = H_{(0,t_n)}$ for $s \in [t_n, t]$.
- (4) There exists $y_i \in C_{(x_{i+1}, t_{i+1})} \cap C_{(x_i, t_i)}$ such that $H_{(y_i, s)} = H_{y_i, t_i}$ for $s \in [t_i, t_{i+1})$ and $i = 0, \dots, (n-1)$.

²There is no direct argument proving that it also holds a.s. here.

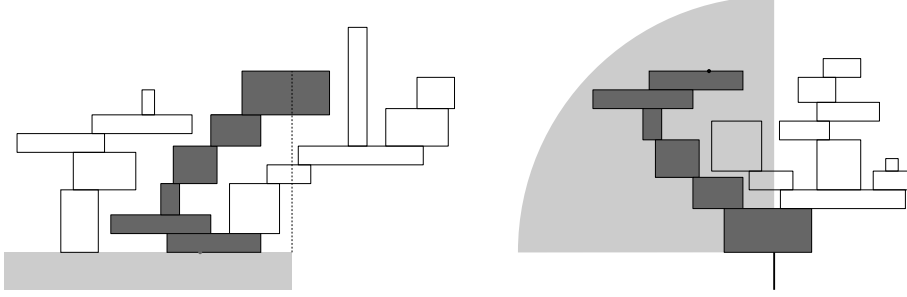


FIGURE 4. Visualization of the "duality argument" in the proof of Theorem 5.1 in the case $x = 0$.

- (5) There exists $z \in C_{(x_0, t_0)} \cap \mathcal{K}$ and $H_{(z, s)} = 0$ for $s \in [0, t_0]$.

Let now $(\tilde{x}_i, \tilde{t}_i) = T_{(0, t)} R(x_{n-i}, t_{n-i}) = (x_{n-i}, t - t_{n-i})$ and $\tilde{y}_i = y_{n-i}$. Then, by the compatibility properties, these quantities satisfy:

- (1) $(\tilde{x}_i, \tilde{t}_i) \in \text{supp } \Phi \circ V \circ S_{(0, t)}$ for $i = 0, \dots, n$.
- (2) $0 \leq t_i < t_{i+1} < t$ for $i = 0, \dots, (n-1)$.
- (3) $z \in C_{(\tilde{x}_n, \tilde{t}_n)}$ and $H_{(z, s)}^{(0)} \circ V \circ S_{(0, t)} = H_{(z, t_n)}^{(0)} \circ V \circ S_{(0, t)}$ for $s \in [\tilde{t}_n, t]$.
- (4) $\tilde{y}_{i+1} \in C_{(\tilde{x}_{i+1}, \tilde{t}_{i+1})} \circ V \circ S_{(0, t)} \cap C_{(\tilde{x}_i, \tilde{t}_i)} \circ V \circ S_{(0, t)}$ such that $H_{(\tilde{y}_i, s)}^{(0)} \circ V \circ S_{(0, t)} = H_{(\tilde{x}_i, \tilde{t}_i)}^{(0)} \circ V \circ S_{(0, t)}$ for $s \in [\tilde{t}_i, \tilde{t}_{i+1})$ and $i = 0, \dots, (n-1)$.
- (5) $0 \in C_{(\tilde{x}_1, \tilde{t}_1)}$ and $H_{(0, s)}^{(0)} \circ V \circ S_{(0, t)} = 0$ for $s \in [0, \tilde{t}_0]$.

Given that $z \in \mathcal{K}$ and \mathcal{K} is the convex cone \mathcal{C} , then for any $h > 0$,

$$\frac{(z, h)}{|(z, h)|} \in \Theta(\mathcal{K}).$$

Then,

$$\begin{aligned} \mathbb{H}_{(0, t)}^{(\Theta(\mathcal{K}), 0)} \circ V \circ S_{(0, t)} &\geq H_{(z, t)}^{(0)} \circ V \circ S_{(0, t)}, \\ &= \sum_{i=0}^n \sigma_{(\tilde{x}_i, \tilde{t}_i)} \circ V \circ S_{(0, t)}, \\ &= \sum_{i=0}^n \sigma_{(x_i, t_i)}, \\ &= H_{(0, t)}. \end{aligned}$$

The proof of the inequality in the other direction is similar to the previous one but starting with the dynamics of $\mathbb{H}_{(0, t)}^{(\Theta(\mathcal{K}), 0)} \circ V \circ S_{(0, t)}$. We omit the details as they are mainly technical. See also Figure 5.

Case 2: $x \in \mathcal{K}$.

The result in this case is obtained by comparison with the growth of the vertex studied in Case 1. From the monotonicity w.r.t. the initial substrates,

$$(5.9) \quad H_{(0,t)}^{(0)} \circ S_{(x,0)} \leq H_{(x,t)}.$$

On the other hand, using $\mathcal{K}^{(1)} = \{0, x\}$, we have the following identity,

$$(5.10) \quad H_{(x,t)} \leq \mathbb{H}_t^{(\Theta(\mathcal{K}),1)} \circ V \circ S_{(0,t)}.$$

To prove the last relation, we use again a set of points $(x_0, t_0), \dots, (x_n, t_n) \in \mathbb{R}^d \times [0, t]$ connecting x with its height. As before:

- (1) $(x_i, t_i) \in \text{supp } \Phi$ for $i = 0, \dots, n$.
- (2) $0 \leq t_i < t_{i+1} < t$ for $i = 0, \dots, (n-1)$.
- (3) $0 \in C_{(x_n, t_n)}$ and $H_{(0,s)} = H_{(0,t_n)}$ for $s \in [t_n, t]$.
- (4) There exists $y_i \in C_{(x_{i+1}, t_{i+1})} \cap C_{(x_i, t_i)}$ such that $H_{(y_i, s)} = H_{y_i, t_i}$ for $s \in [t_i, t_{i+1})$ and $i = 0, \dots, (n-1)$.
- (5) There exists $z \in C_{(x_0, t_0)} \cap \mathcal{K}$ and $H_{(z, s)} = 0$ for $s \in [0, t_0]$.

When we consider now $(\tilde{x}_i, \tilde{t}_i) = T_{(0,t)} \circ R(x_{n-i}, t_{n-i}) = (x_{n-i}, t - t_{n-i})$ and $\tilde{y}_i = y_{n-i}$, we then have that there exists a path of RACS starting at x and finishing at $(z, H_{(x,t)}) \in \Theta(\mathcal{K})$. By the very definition of $\mathbb{H}_t^{(\Theta(\mathcal{K}),1)} \circ V \circ S_{(0,t)}$ this then implies (5.10).

The desired limit then follows from (5.9) and (5.10), the result of Case 1 and Lemma 4.2. Notice that the L_1 limit is hence the same for all points $x \in \mathcal{K}$.

Case 3: $x \notin \mathcal{K}$. Let $x \notin \mathcal{K}$ and let u be the Euclidean distance from x to \mathcal{K} , so $|x - y| = u$ for some $y \in \mathcal{K}$. On the segment $[x, y]$, choose points $x_0 = y, x_1, \dots, x_m = x$ equidistantly, where m is the smallest integer that exceeds $2u/r$. Consider shifted versions of \mathcal{K} , say $\mathcal{K}_0 = \mathcal{K}, \mathcal{K}_1, \dots, \mathcal{K}_m$ such that for any $i \geq 1$, $\mathcal{K}_i \supset \mathcal{K}_{i-1}$, and \mathcal{K}_i includes the points x_0, \dots, x_i and does not include the points x_{i+1}, \dots, x_m . Then we show the convergence of $\max(0, H_{(x_i, t)})$ to Z in L_1 using an induction argument: if the convergence holds for x_{i-1} , then it also holds for x_i . Because of that, we may assume without loss of generality that $m = 1$, so $y = x_0$, $x = x_1$ and $|x - y| \leq r/2$.

Let $\tilde{\mathcal{K}} = \mathcal{K}_1$ and let $\tilde{H}_{(y,t)}$ be the height associated with the cone $\tilde{\mathcal{K}}$. Let ε be a positive number.

First, we show that

$$(5.11) \quad \lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{H_{(y,t)}}{t} > Z + \varepsilon \right) = 0$$

and that the random variables $\max(0, H(y, t))/t$ are uniformly integrable. By monotonicity (see Subsection 2.3), we have $H_{(y,t)} \leq \tilde{H}_{(y,t)}$ and, in view of

the previous cases, $\max(0, \tilde{H}_{(y,t)})/t \rightarrow Z$ in L_1 and, therefore, in probability. Therefore, both (5.11) and uniform integrability of $H_{(y,t)}/t$ follow.

Secondly, we show that

$$(5.12) \quad \lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{H_{(y,t)}}{t} < Z - \varepsilon \right) = 0.$$

Indeed, let $\Pi_{x,y}$ be a stream of RACS's that contain a ball of radius r that covers both x and y . By our assumptions, this is a homogeneous Poisson process of positive intensity, say ν . For each t , let $t - \eta_t$ be the last arrival of such a RACS before t . Clearly, the random variable η_t has an exponential distribution with parameter ν . Further, $H_{(y,t)} \geq H_{(x,t-\eta_t)}$ a.s., so for any $T > 0$,

$$\mathbb{P} \left(\frac{H_{(y,t)}}{t} < Z - \varepsilon \right) \leq \mathbb{P}(\eta_t > T) + \mathbb{P} \left(\frac{H_{(x,t-T)}}{t} < Z - \varepsilon \right) \rightarrow e^{-\nu T}$$

as $t \rightarrow \infty$. Letting $T \rightarrow \infty$ leads to (5.12).

Finally equations (5.11) and (5.12) imply the convergence in probability $H_{(x,t)}/t \rightarrow Z$ and, further, uniform integrability implies the L_1 -convergence of $\max(0, H_{(x,t)})$ to Z .

□

Definition 5.2. We say that $\mathcal{K} \subseteq \mathbb{R}^d$ is similar to the closed convex cone $\mathcal{K}^{(c)} \subseteq \mathbb{R}^d$ if there exists two vectors $v_{\pm} \in \mathbb{R}^d$ such that

$$\mathcal{K}^{(c)} + v_- \subseteq \mathcal{K} \subseteq \mathcal{K}^{(c)} + v_+.$$

Remark 5.2. Notice that if \mathcal{K} is a convex cone, then it is trivially similar to itself. Also, if \mathcal{K} is similar to the convex cones $\mathcal{K}_1^{(c)}$ and $\mathcal{K}_2^{(c)}$ then $\mathcal{K}_1^{(c)} = \mathcal{K}_2^{(c)}$ by the geometry of the convex cones.

By monotonicity we obtain the following corollary from Theorem 5.1

Corollary 5.3. Assume the hypothesis of Corollary 3.9 holds. Given $\mathcal{K} \subseteq \mathbb{R}^d$ similar to a closed convex cone $\mathcal{K}^{(c)}$ with vertex at the origin, for all $x \in \mathcal{K}$,

$$\lim_{t \rightarrow \infty} \frac{\max(H_{(x,t)}, 0)}{t} = \lim_{t \rightarrow \infty} \frac{\mathbb{E} \max(H_{(x,t)}, 0)}{t} = \sup_{t > 0} \frac{\mathbb{E} \mathbb{H}_t^{(\Theta(\mathcal{K}^{(c)}), 0)}}{t} < \infty,$$

where the first limit is in L_1 .

6. APPENDIX

Proof. of Lemma 3.3.

The proof leverages the ideas developed in the proof of Theorem 2 in [2]. We use the same discretization of time and space as in the proof of this theorem to show the following:

- (1) There exists a branching process constructed from an i.i.d. family of random variables $\{(v_i, s_i)\}_i$ with light-tails. For a given i , v_i denotes the number of offsprings of i and s_i denotes the (common) *height* of its offspring.
- (2) For $n \in \mathbb{N}$, let $h(n)$ denote the maximum height of this branching process at generation n , namely the maximum, over all lineages, of the sum of the heights of all generations in the lineage. Then, in order to prove (3.3), it suffices to prove that $\mathbb{E}h(n) \leq Cn$ for every $n > 0$ for some finite C .

For $n \in \mathbb{N}$, let d_n denote the number of individuals of generation n in this branching process. For $a > 0$, let

$$D(a) := \bigcup_{n \geq 1} \{d_n > a^n\}, \quad \bar{D}(a) := \Omega \setminus D(a).$$

Let $a_m = Cm$, $m \in \mathbb{N}$. From Chernoff's inequality, for some sufficiently large constant $C > 0$,

$$\mathbb{P}(D(a_m)) \leq 2^{-m}.$$

From Chernoff's inequality again, we get that for all $i \in \mathbb{N}$, $\delta > 0$ and $c_m > 0$ to be fixed we get,

$$\mathbb{P}\left(\frac{h(n)}{n} > (c_m + i) \cap \bar{D}(a_m)\right) \leq \left(a_m \mathbb{E}(e^{\delta s}) e^{-\delta c_m}\right)^n e^{-\delta n i},$$

where s is a typical height. Therefore,

$$\begin{aligned} \mathbb{E}\left(\frac{h(n)}{n}\right) &= \sum_{m \geq 1} \mathbb{E}\left(\frac{h(n)}{n} 1_{\bar{D}(a_m) \setminus \bar{D}(a_{m-1})}\right), \\ &\leq \sum_{m \geq 1} \left(\sum_{i \geq 0} \mathbb{P}\left(\frac{h(n)}{n} > (c_m + i) \cap \bar{D}(a_m)\right) \right) + \mathbb{P}(\bar{D}(a_{m-1})) c_m, \\ &\leq \sum_{m \geq 1} \left(\frac{a_m \mathbb{E}(e^{\delta s}) e^{-\delta c_m}}{1 - e^{-\delta}} \right)^n + 2(2^{-m} c_m). \end{aligned}$$

Now we fix δ sufficiently small such that $\mathbb{E}(e^{\delta s}) < \infty$. Recalling that $a_m = Cm$, in order to conclude the proof, it suffices to construct c_m independent of n , such that

$$\left(Cm e^{-\delta c_m}\right)^n \leq 2^{-m}, \quad \sum_{m \geq 1} 2^{-m} c_m < \infty,$$

where C is a constant independent of n . The last bound is satisfied for $c_m = Bm$ for any $B > 0$. However, for B sufficiently large $Ce^{-\delta Bm} \leq 4^{-m}$ which concludes the proof. \square

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